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# Asymptotic Completeness of Certain Four-Body Schrödinger Operators

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We give a proof of asymptotic completeness for four-body Schrödinger operators. The two-body potentials are assumed to be short range and there are spectral assumptions on the two- and three-body subsystems. These spectral assumptions hold generically for certain classes of potentials. The proof of the main theorem depends on an analysis of the rates of decay of the wave function in certain regions of configuration space, depending on the scattering channel to which the wave function belongs. © 1986 Academic Press, Inc.

## 1. INTRODUCTION

In this paper we study scattering theory for certain Schrödinger operators which describe the quantum mechanics of four particles moving in $n \geqslant 3$ dimensions. Our main result is a theorem on asymptotic completeness, which is essentially the same as the main result of [10]. However, the proof we present here is much simpler and contains much more physical intuition than the one in [10].

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The Schrödinger operator for $N$ particles moving in $n$ dimensions is $\tilde{H} = -\sum_{i=1}^{N} (2m_i)^{-1} \Delta_i + \sum_{i<j} V_{ij}$ on $L^2(\mathbb{R}^{Nn})$. Here the mass of the $i$th particle is $m_i$, $\Delta_i$ is the $n$-dimensional Laplacian in the $i$th variable, and the potential energy between particles $i$ and $j$ is the multiplication operator $V_{ij}(x_i - x_j)$. By removing the trivial center of mass motion for $\tilde{H}$ (see [19]) we obtain $\tilde{H} = H_0 + \sum_{i<j} V_{ij}$ on $\mathscr{H} = L^2(\mathbb{R}^{(N-1)n})$.

The scattering theory for $H$ involves the asymptotic behavior of solutions to the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = H\psi \tag{1.1}$$

as $t \to \pm\infty$. If the potentials $V_{ij}(x_i - x_j)$ decrease sufficiently rapidly as $|x_i - x_j| \to \infty$, then from physical arguments one expects all solutions to (1.1) to be superpositions of solutions of the following types:

1. Bound state solutions, i.e., $e^{-itE}\psi$, where $H\psi = E\psi$ and $\psi \in \mathscr{H}$.

2. Solutions which asymptotically describe $k$ bound clusters of particles, with the clusters asymptotically moving freely relative to one another. Here $k$ may take on values 2, 3, 4,..., $N$.

The asymptotic completeness problem is to prove that this physical picture is correct.

To formulate the problem precisely we need some notation. A *cluster decomposition* $D = \{C_i\}_{i=1}^{k}$ is a partition of the set $\{1, 2, \ldots, N\}$ into $k$ disjoint *clusters* $C_i$. If $D$ has $k$ clusters we sometimes denote $D$ as $D_k$. We define $H_D = H_0 + V_D$, where $V_D$ is the sum of all $V_{ij}$ with $i$ and $j$ in the same cluster. For each $D$, the Hilbert space can be decomposed as $\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_k \otimes \mathscr{H}(D)$, so that

$$\begin{aligned} H_D = {} & h_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \\ & + 1 \otimes h_2 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \\ & + \cdots \\ & + 1 \otimes 1 \otimes 1 \otimes \cdots \otimes h_k \otimes 1 \\ & + 1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes K_D. \end{aligned}$$

The *cluster Hamiltonian* $h_i$ corresponds to the energy of the particles in cluster $C_i$ alone. $K_D$ is the kinetic energy of the centers of mass of the clusters in $D$.

For each $i = 1, 2, \ldots, k$ we choose eigenfunctions $\eta_{i,j} \in \mathscr{H}_i$ of $h_i$ so that $\{\eta_{i,j}\}$ is an orthonormal basis for the subspace of $\mathscr{H}_i$ spanned by the eigenfunctions of $h_i$. A *channel* $\alpha$ is a cluster decomposition $D(\alpha)$ together with an eigenfunction $\eta_i \in \{\eta_{i,j}\}$ for each $h_i$. We define the *threshold* associated with $\alpha$ to be the number $E_\alpha = \sum_{i=1}^{k} E_i$, where $h_i \eta_i = E_i \eta_i$, and let $P_\alpha: \mathscr{H} \to \mathscr{H}$ denote the orthogonal projection onto all vectors of the form $\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_k \otimes \phi$, where $\phi \in \mathscr{H}(D(\alpha))$ is arbitrary. Let $T_\alpha = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes (K_{D(\alpha)} + E_\alpha)$ so that $H_{D(\alpha)} P_\alpha = T_\alpha P_\alpha$, and let $P_D$ be the projection obtained by summing all $P_\alpha$ with $D(\alpha) = D$.

For all the potentials $V_{ij}$ which we will consider, the *channel wave operators* $\Omega_\alpha^\pm = \text{strong-lim}_{t \to \mp\infty} e^{itH} e^{-itT_\alpha} P_\alpha$ exist (see [19, 23]). A state $\psi \in \mathscr{H}$ which belongs to the range of $\Omega_\alpha^-$ corresponds to a $\phi \in \mathscr{H}(D(\alpha))$ such that $e^{-itH}\psi$ is asymptotic to $e^{-itT_\alpha}(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_k \otimes \phi)$. Thus, as $t \to \infty$, $e^{-itH}\psi$ evolves asymptotically as $k$ freely moving clusters of particles with each cluster in a bound state.

The collection of all channels describes all the different ways in which the particles can move to infinity as such freely moving bound clusters. Since the ranges of the channel wave operators are orthogonal (see [19]), our physical intuition suggests that $\mathscr{H} = \bigoplus_\alpha \operatorname{Ran} \Omega_\alpha^\pm \oplus \mathscr{H}_{\text{bound}}$. If this is true, then the scattering for $H$ is *asymptotically complete.*

Our main result is the following:

THEOREM 1.1. *Let $n \geqslant 3$ and $N \leqslant 4$. Let $H = H_0 + \sum_{i<j} V_{ij}$, on $L^2(\mathbb{R}^{(N-1)n})$, where each $V_{ij}$ has the form $V_{ij} = (1 + x_{ij}^2)^{-\gamma} Y_{ij}(x_{ij})$, where $\gamma > 1$ and $Y_{ij} \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for some $p > n/2$. Suppose the cluster Hamiltonians for the subsystems of H have no eigenvalues embedded in their continuous spectra, no threshold eigenvalues, and no threshold resonances. Then asymptotic completeness holds for H.*

For the precise meaning of the spectral hypotheses on the subsystems, see Section 5.

One expects these hypotheses to hold for almost all potentials. In particular with a dilation analyticity assumption (see [2, 10, 20, 22]) the following is true.

THEOREM 1.2. *Let $n \geqslant 3$ and $N \leqslant 4$. Let $H = H_0 + \sum_{i<j} \lambda_{ij} V_{ij}$ on $L^2(\mathbb{R}^{(N-1)n})$. Assume each $V_{ij}$ may be factored as $V_{ij} = U_{ij} W_{ij}$ so that*

(i) *each $U_{ij}$ and $W_{ij}$ is a dilation analytic in some strip;*

(ii) *$(1 + x_{ij}^2)^\delta U_{ij}(x_{ij})$ and $(1 + x_{ij}^2)^\delta W_{ij}$ belong to $L^q(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for some $\delta > \frac{1}{2}$ and $q > n$.*

*Then there exists a closed set $\mathscr{E} \in \mathbb{R}^{N(N-1)/2}$ of measure zero with the property that $\{\lambda_{ij}\} \notin \mathscr{E}$ implies asymptotic completeness for H.*

This result is slightly stronger than that of [10]. The main theorem of [10] requires the additional hypothesis that the bound states of three-body cluster Hamiltonians have non-positive energy when $N = 4$. (However, Herbst and Froese [8] have recently proved this condition holds for a large class of potentials.)

We will only prove Theorem 1.1. Furthermore, in order to shorten the exposition we will only prove Theorem 1.1 under the additional assumption that all potentials are bounded. The local singularities can be handled by the methods mentioned in [11]. The proof of Theorem 1.2 is essentially the same as that of Theorem 1.1 except that one must prove the exceptional set $\mathscr{E}$ is closed and has measure zero. One does this by mimicking Section VI of [10], using the equations from Section 5 of the present paper in place of those of [10]. Since this is so similar to the argument in [10], we have omitted the proof here.

As mentioned above, our results are not particularly new although our proof is. The principal ideas are the same as those used in our previous paper [11] to deal with three bodies. The intuition is time dependent, but it is somewhat easier to present the details in a time independent setting.

V. Enss [5, 6] recently gave a general proof of three-body asymptotic completeness. This proof involves a very clever analysis of the long time behavior of the three-body system. In contrast, with the exception of [5, 6, 11, 16], other results which deal with the multichannel problem prove asymptotic completeness by time independent methods (see [7, 9, 10, 12–15, 21, 24, 25]). These methods involve a more or less explicit construction of the wave operators in terms of the resolvent of the Hamiltonian. This approach obscures physical intuition, although in principle it gives explicit formulas for the wave operators. Faddeev [7] first used this idea to treat generic three-body problems. His proof has been clarified and simplified in [9, 13, 24, 25]. Hagedorn [10] extended the proof to include four bodies; Sigal [21] used the same ideas but different resolvent equations to deal with generic $N$-body systems. Loss and Sigal [14] have also given an extension of the time independent method to handle some non-generic three-body systems.

Our present proof of four-body asymptotic completeness can be phrased in a completely time dependent fashion as was done for three bodies in [11]. Since it is simpler to do some steps in a time independent way, we have done so, but we urge the reader to keep in mind the time dependent intuition.

The intuition goes as follows: If $\psi_1$ is orthogonal to the bound states, then $\rho(x)(e^{-itH}\psi_1)(x)$ should tend to zero in an $L^2(dx)$ sense as $t \to \infty$ for any function $\rho$ which is concentrated in a region of configuration space in which all four particles must be near one another. A local Kato smoothness estimate of Perry, Sigal, and Simon [18] shows that this is true in the sense that

$$\int_{-\infty}^{\infty} \|\rho(\cdot)(e^{-itH}\psi_1)(\cdot)\|^2_{L^2(dx)}\, dt \tag{1.1}$$

is finite for all $\psi_1$ in a dense subset of the orthogonal complement of the bound states. This estimate implies "two cluster completeness," i.e., the strong limits

$$P_\alpha^\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} P_\alpha e^{-itH} (1 - P_{\text{bound}})$$

exist for two cluster channels $\alpha$, and $P_\alpha^\pm$ is the orthogonal projection onto the range of $\Omega_\alpha^\mp$.

Next let the cluster decomposition $D_2$ have exactly two clusters. Suppose $\psi_2$ is orthogonal to the bound states and orthogonal to the ranges of all $\Omega_\alpha^-$ with $D(\alpha) = D_2$. Let $\rho_{D_2}$ be a function which is supported in a region of configuration space in which the particles in the same clusters of $D_2$ must be near one another. Then physically one expects $\rho_{D_2} e^{-itH} \psi_2$ to tend to zero as $t \to \infty$. We prove this in the sense that

$$\int_0^\infty \|\rho_{D_2} e^{-itH} \psi_2\|^2 \, dt \tag{1.2}$$

is finite for a dense set of $\psi_2$'s in the orthogonal complement of the subspace spanned by the bound states and the ranges of the $\Omega_\alpha^-$ with $D(\alpha) = D_2$. From these results and the corresponding ones for $t \to -\infty$ for all two cluster $D$'s we prove "three cluster completeness," i.e., the strong limits

$$P_\alpha^\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} P_\alpha e^{-itH} \left(1 - \sum_{\beta_2} P_{\beta_2}^\pm\right) (1 - P_{\text{bound}})$$

exist for all three cluster channels $\alpha$, and $P_\alpha^\pm$ is the orthogonal projection onto the range of $\Omega_\alpha^\mp$. Here the $\sum_{\beta_2}$ denotes the sum over all two cluster channels.

Next suppose $D_3$ has three clusters, and suppose $\psi_3$ is orthogonal to the bound states, orthogonal to the ranges of $\Omega_\beta^-$ for all two cluster channels $\beta$ with $D_3$ a refinement of $D(\beta)$, and orthogonal to the ranges of all $\Omega_\alpha^-$ for all three cluster channels with $D(\alpha) = D_3$. Let $\rho_{D_3}$ be a function which is concentrated in the region of configuration space in which the particles in the clusters of $D_3$ must be near one another. We prove (for a dense set of such $\psi_3$'s) that

$$\int_0^\infty \|\rho_{D_3} e^{-itH} \psi_3\|^2 \, dt \tag{1.3}$$

is finite. From this we are able to conclude that the orthogonal projection onto the range of the four cluster wave operator $\Omega_0^-$ is $(1 - \sum_{\alpha_3} P_{\alpha_3}^+ - \sum_{\alpha_2} P_{\alpha_2}^+ - P_{\text{bound}})$. Here $\sum_{\alpha_2}$ denotes the sum over all two cluster channels. The orthogonal projection onto the range of $\Omega_0^+$ is similarly identified as $(1-\sum_{\alpha_3} P_{\alpha_3}^- - \sum_{\alpha_2} P_{\alpha_2}^- - P_{\text{bound}})$. Asymptotic completeness immediately follows.

Proof of the boundedness of quantities like (1.1), (1.2), and (1.3) is accomplished by solving a system of coupled equations involving the operators inside the norms of (1.1), (1.2), and (1.3) (with projections inserted to keep the vectors in the correct subspaces). The system of equations is derived as a four-body multiple collision expansion. We urge the reader to look at the three-body analog, [11], for motivation since the four-body equations are much more complicated. This complexity reflects the intricate structure which a four-body collision may possess.

The paper is organized as follows: In Section 2 we reduce the proof of Theorem 1.1 to the study of quantities like (1.1), (1.2), and (1.3). In Section 3 we restate the results of Section 2 in a way which is technically more useful but intuitively less appealing. In Section 4 we further reduce the proof of Theorem 1.1 to the study of the behavior of certain operator valued functions near the spectrum of $H$. Section 5 is devoted to the study of generic behavior of subsystem revolvents near thresholds. The proof of Theorem 1.1 is then completed in Section 6.

## 2. Reduction of Asymptotic Completeness to Some Kato Smoothness Estimates

In this section we reduce the proof of Theorem 1.1 to the physically motivated estimates mentioned in the Introduction. We begin by recalling that a closed operator $B$ with domain $\mathscr{D}(B) \subset \mathscr{H}$ is called Kato smooth with respect to a self-adjoint operator $A$ on $\mathscr{H}$ (or equivalently $B$ is $A$-smooth) if for all $\psi \in \mathscr{H}$, $e^{-itA}\psi \in \mathscr{D}(B)$ for a.e.t, and

$$\sup_{\|\psi\|=1} \int_{-\infty}^{\infty} \|Be^{-itA}\psi\|^2\, dt < \infty.$$

An operator $B$ is locally $A$-smooth on a Borel set $\Omega \subset \mathbb{R}$ if $BE_\Omega(A)$ is $A$-smooth. Here $E_\Omega(A)$ is the spectral projection for $A$ onto $\Omega$. There are several equivalent definitions of Kato smoothness which can be found along with much discussion in [20].

In order to state the main result, Proposition 2.1, we need to introduce clustered Jacobi coordinates (see [20]). To obtain Jacobi coodinates $\{\zeta_i\}$, one defines $\zeta_j$ to be the vector from the center of mass of particles $1, 2, 3, \ldots, j$ to particle $j+1$, where $1 \leqslant j \leqslant N-1$. Explicitly, $\zeta_j = x_{j+1} - (\sum_{i \leqslant j} m_i)^{-1} \sum_{i \leqslant j} m_i x_i$, where $x_j$ denotes the position of the $j$th particle. To obtain clustered Jacobi coordinates for a cluster decomposition

$D = \{C_i\}$, one chooses Jacobi coordinates $\xi_1^i, \xi_2^i, ..., \xi_{n(i)-1}^i$ for the $n(i)$ particles in the $i$th cluster. One then chooses $\zeta_1, \zeta_2, ..., \zeta_{k-1}$ to be Jacobi coordinates for the centers of mass of the clusters. The collection $\xi_1^1, \xi_2^1, ..., \xi_{n(1)-1}^1, \xi_1^2, ..., \xi_{n(k-1)}^k, \zeta_1, ..., \zeta_k$ are the clustered Jacobi coordinates. In such a coordinate system, the operator $K_D$ has the form $K_D = -\sum_{i=1}^{k-1} (2\mu_i)^{-1} \Delta_{\zeta_i}$.

Certain functions $\rho_D(x)$ which depend on the intracluster coordinates $\xi_i^j$ play a special role in the proof of Theorem 1.1. Suppose $\eta > 1$ is chosen so that all the values of $\gamma$ which occur in the statement of Theorem 1.1 are greater than $\eta$. For each cluster decomposition $D$ choose a clustered Jacobi coordinate system. Then $\rho_D$ is defined as $\rho_D(x) = (1 + \sum c_{ij}(\xi_i^j)^2)^{-\eta/2}$, where the sum runs over all intracluster coordinates and the $c_{ij}$ are the appropriate reduced masses. With the proper choices of the $c_{ij}$'s the $\rho_D$'s are uniquely defined, independent of the particular choices of clustered Jacobi coordinates.

We can now state the main result of this section.

PROPOSITION 2.1. *Let $H$ satisfy the hypotheses of Theorem* 1.1, *and assume that the potentials $V_{ij}$ are bounded. Suppose there exists a closed set $S_{\mathrm{ex}}$ of measure zero such that all operators of the following types are locally $H$-smooth on compact subintervals of $\sigma_{\mathrm{a.c.}}(H) \backslash S_{\mathrm{ex}}$:*

(i) $\rho_{D_1}$, *where $D_1 = \{1, 2, 3, 4\}$ has one cluster;*

(ii) $\rho_{D_2}(1 - P_{D_2})$, *where $D_2$ has two clusters;*

(iii) $\rho_{D_3}(1 - P_{D_3})(1 - \sum_{D_2 \supset D_3} P_{D_2})$, *where $D_3$ has three clusters and the sum is over all two cluster $D_2$'s which are refined by $D_3$.*

*Then asymptotic completeness holds.*

*Remarks* 1. The exceptional set $S_{\mathrm{ex}}$ is the set of thresholds and bound states, and is at most countable (see Section 6 and [18]).

2. The fact that $\rho_{D_1}$ is locally $H$-smooth has been proved in [18]. The conditions (ii) and (iii) are verified in Sections 4, 5, and 6.

We have broken the proof of this proposition into five lemmas. The first gives an abstract result which is used in the last three lemmas. The second proves a technical estimate. The third, fourth and fifth prove two cluster, three cluster, and four cluster completeness, respectively.

LEMMA 2.2. *Let $A_1$ and $A_2$ be self-adjoint operators on a Hilbert space, and let $J$ be a bounded operator on that space. Suppose that the quadratic form on $\mathscr{D}(A_1) \times \mathscr{D}(A_2)$ associated with $A_1 J - J A_2$ can be factored as*

$A_1J - JA_2 = \sum_{i=1}^{N} B_i^* C_i D_i$, *where* $B_i$, $C_i$, *and* $D_i$ *satisfy the following conditions*:

(1) *Each* $B_i$ *is a* $A_1$*-smooth.*

(2) *Each* $C_i$ *is bounded.*

(3) *There exists a closed set* $M \subset \mathbb{R}$ *of measure zero such that each* $D_i$ *is locally* $A_2$*-smooth on compact subintervals of* $\sigma_{\text{a.c.}}(A_2)\backslash M$.

*Then* $W^{\pm}(A_1, A_2; J) = \text{s-lim}_{t \to \pm\infty} e^{itA_1} J e^{-itA_2} P_{\text{a.c.}}(A_2)$ *exist, where* $P_{\text{a.c.}}(A_2)$ *denotes the absolutely continuous spectral projection for* $A_2$.

*Proof.* Combine the proof of Theorem 1.1 of [11] with Remark 1.2 of [11]. ∎

LEMMA 2.3. (a) *Assume the hypotheses of Theorem* 1.1, *and assume the potentials* $V_{ij}$ *are bounded. Let* $D$ *be any cluster decomposition with at least two clusters, and suppose* $i$ *and* $j$ *belong to different clusters of* $D$. *Let* $D'$ *be the cluster decomposition obtained from* $D$ *by merging the clusters which contain* $i$ *and* $j$. *Then*

$$\rho_{D'}^{-1} |V_{ij}|^{1/2} P_D, \qquad \rho_{D'}^{-1} P_D |V_{ij}|^{1/2}, \qquad \textit{and} \qquad \rho_{D'}^{-1} P_D V_{ij} \rho_{D'}^{-1}$$

*are bounded operators on* $\mathscr{H}$.

(b) *If* $D$ *and* $D'$ *are non-trivial cluster decompositions and* $D''$ *is the cluster decomposition with the greatest number of clusters which is refined by* $D$ *and* $D'$, *then* $\rho_{D''}^{-1} P_D \rho_{D'}$ *and* $\rho_{D''}^{-1} \rho_{D'} P_D$ *are bounded operators on* $\mathscr{H}$.

*Proof.* The assumptions of Theorem 1.1 imply the exponential fall off of eigenfunctions of subsystem Hamiltonians [3, 4, 17, 20]. The lemma follows from this fact and estimates involving different systems of clustered Jacobi coordinates. See Lemma V.4 of [10] and Proposition 2.3 of [9]. ∎

*Remark.* If $\alpha$ is a channel with cluster decomposition $D(\alpha)$, then the results in Lemma 2.3 are still true if $P_D$ is replaced by $P_\alpha$. Of course, in (a) one must choose $D'$ as described in the lemma with $D = D(\alpha)$, and in (b) one must choose $D''$ as described in the lemma with $D = D''$. The proof of this is the same as the proof of the lemma.

LEMMA 2.4. *Under the conditions of Proposition* 2.1 *the following strong limits exist when* $\alpha$ *is a two cluster channel*:

(i) $\Omega_\alpha^{\pm} = \text{s-lim}_{t \to \mp\infty} e^{itH} e^{-itT_\alpha} P_\alpha$,

(ii) $W_\alpha^{\pm} = \text{s-lim}_{t \to \pm\infty} e^{itT_\alpha} P_\alpha e^{-itH} P_{\text{a.c.}}$,

(iii) $P_\alpha^{\pm} = \text{s-lim}_{t \to \pm\infty} e^{itH} P_\alpha e^{-itH} P_{\text{a.c.}}$.

*(Here $P_{\text{a.c.}}$ denotes the absolutely continuous projection for $H$.) Furthermore, the adjoint of $\Omega_\alpha^\mp$ is $W_\alpha^\pm$, and $P_\alpha^\pm$ is the orthogonal projection onto the range of $\Omega_\alpha^\mp$.*

*Proof.* The existence of the limits defining the wave operators $\Omega_\alpha^\pm$ is well known [19, 23].

Next, let $H_\alpha = H_{D(\alpha)}$, and note that $H_\alpha P_\alpha = T_\alpha P_\alpha$. Thus $W_\alpha^\pm =$ s-$\lim_{t\to\pm\infty} e^{itH_\alpha} P_\alpha e^{-itH} P_{\text{a.c.}}$ if the limits exist. To see that these limits exist we need only show that Lemma 2.2 can be applied. However, $H_\alpha P_\alpha - P_\alpha H = -P_\alpha I_{D(\alpha)}$, where $I_{D(\alpha)}$ is the sum of all $V_{ij}$ with $i$ and $j$ in different clusters of $D(\alpha)$. For such $i$ and $j$ we can factor $P_\alpha V_{ij}$ as

$$P_\alpha V_{ij} = [P_\alpha(1+x_{ij}^2)^{-\eta/2}][(1+x_{ij}^2)^{\eta/2} P_\alpha V_{ij} \rho_{D_1}^{-1}]\, \rho_{D_1}, \tag{2.1}$$

where $\eta$ is chosen as in the definition of the $\rho$ functions. A standard argument shows that the adjoint of the first factor is $H_\alpha$-smooth (see the proof of Lemma II.3 of [10]). The second factor is bounded by Lemma 2.3 and the choice of $\eta$. The third factor satisfies condition (3) of Lemma 2.2 by hypothesis with $M = S_{\text{ex}}$. The existence of the limits defining $W_\alpha^\pm$ now follows from Lemma 2.2.

The existence of the limits defining $P_\alpha^\pm$ now follows by writing

$$\begin{aligned}
P_\alpha^\pm &= \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} P_\alpha e^{-itH} P_{\text{a.c.}} \\
&= \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} P_\alpha e^{-itT_\alpha} e^{itT_\alpha} P_\alpha e^{-itH} P_{\text{a.c.}} \\
&= (\operatorname*{s-lim}_{t\to\pm\infty} e^{itH} P_\alpha e^{-itT_\alpha})(\operatorname*{s-lim}_{t\to\pm\infty} e^{iT_\alpha} P_\alpha e^{-itH} P_{\text{a.c.}}) \\
&= \Omega_\alpha^\mp W_\alpha^\pm .
\end{aligned}$$

Similar calculations show that $P_\alpha^\pm$ is a projection and that $P_\alpha^\pm = W_\alpha^\pm \Omega_\alpha^\mp$ (here one needs to know that $\operatorname{Ran} \Omega_\alpha^\pm \subset P_{\text{a.c.}} \mathscr{H}$; see Section XI.3 of [19]). The lemma now follows by some simple calculations since the $\Omega_\alpha^\pm$ are partial isometries. ∎

LEMMA 2.5. *Assume the hypotheses of Proposition* 2.1, *and let $\alpha$ be a three cluster channel. The following strong limits exist*:

(i) $\Omega_\alpha^\pm =$ s-$\lim_{t\to\mp\infty} e^{itH} P_\alpha e^{-itT_\alpha}$,

(ii) $W_\alpha^\pm =$ s-$\lim_{t\to\pm\infty} e^{itT_\alpha} P_\alpha e^{-itH}(1 - \sum_{\beta_2} P_{\beta_2}^\pm)\, P_{\text{a.c.}}$,

(iii) $P_\alpha^\pm =$ s-$\lim_{t\to\pm\infty} e^{itH} P_\alpha e^{-itH}(1 - \sum_{\beta_2} P_{\beta_2}^\pm)\, P_{\text{a.c.}}$.

*(The sums in* (ii) *and* (iii) *denote the sum of all $P_\beta^\pm$ which were constructed in Lemma* 2.4.) *The adjoint of $\Omega_\alpha^\mp$ is $W_\alpha^\pm$ and $P_\alpha^\pm$ is the orthogonal projection onto the range of $\Omega_\alpha^\mp$.*

*Proof.* The existence of the limits defining the wave operators $\Omega_\alpha^\pm$ is well known [19, 23].

If $\psi \in \mathscr{H}$, then $\|e^{-itH}(1-\sum_{\beta_2} P_{\beta_2}^\pm)\psi - (1-\sum_{\beta_2} P_{\beta_2})\, e^{-itH}\psi\|$ tends to zero as $t \to \pm\infty$. Thus, the existence of the limits in (ii) is equivalent to the existence of

$$\text{(ii)}' \quad W_\alpha^\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itT_\alpha} P_\alpha \left(1 - \sum_{\beta_2} P_{\beta_2}\right) e^{-itH} P_{\text{a.c.}}.$$

To prove existence of these limits we mimic the proof of Lemma 2.4 and consider $H_\alpha P_\alpha(1-\sum_{\beta_2} P_{\beta_2}) - P_\alpha(1-\sum_{\beta_2} P_{\beta_2})H$. This operator may be written as a sum of two terms

$$P_\alpha \left[\left(1 - \sum_{\beta_2} P_{\beta_2}\right), H\right] \tag{2.2}$$

and

$$P_\alpha I_\alpha \left(1 - \sum_{\beta_2} P_{\beta_2}\right), \tag{2.3}$$

where $I_\alpha$ denotes the sum of all $V_{ij}$ with $i$ and $j$ in different clusters of $D(\alpha)$.

The operator in (2.2) can be written as

$$P_\alpha \sum_{\beta_2} [H, P_{\beta_2}],$$

and $[H, P_{\beta_2}] = [I_{\beta_2}, P_{\beta_2}]$. By a bound similar to that used in Lemma 2.3, $\rho_{D_1}^{-1}[I_{\beta_2}, P_{\beta_2}]\,\rho_{D_1}^{-1}$ is bounded since $\beta_2$ is a two cluster channel. The operator $\rho_{D_1}^{-1} P_\alpha \rho_{D_1}$ is bounded by the remark after Lemma 2.3. So, operators of the form (2.2) can be written as sums of terms of the form

$$\rho_{D_1}\,(\text{bounded operator})\,\rho_{D_1}.$$

The operator $\rho_{D_1}$ is $H_\alpha$-smooth by the argument in the proof of Lemma 2.4 and it is locally $H$-smooth on compact subintervals of $\sigma_{\text{a.c.}}(H)\backslash S_{\text{ex}}$ by the hypothesis of Proposition 2.1. Thus, terms of the form (2.2) can be factored in a way which satisfies the hypotheses of Lemma 2.2.

Next consider operators of the form (2.3). If $i$ and $j$ belong to different clusters of $D(\alpha)$, then $\rho_{D_2}^{-1} P_\alpha V_{ij} \rho_{D_2}^{-1}$ is bounded by Lemma 2.3. Here $D_2$ is obtained from $D(\alpha)$ by merging the clusters containing $i$ and $j$.

The adjoint of $P_\alpha \rho_{D_2}$ is $H_\alpha$-smooth by the argument used to prove the smoothness of the adjoint of the first factor of (2.1). So, in order to apply Lemma 2.2 we need only show $\rho_{D_2}(1-\sum_{\beta_2} P_{\beta_2})$ is locally $H$-smooth on compact subintervals of $\sigma_{\text{a.c.}}(H)\backslash S_{\text{ex}}$.

To show this we write $\rho_{D_2}(1-\sum_{\beta_2} P_\beta)$ as $\rho_{D_2}(1-P_{D_2})-\sum_{\beta_2, D(\beta_2)\neq D_2}\rho_{D_2} P_{\beta_2}$. The first of these terms has the desired local smoothness by hypothesis (ii) of Proposition 2.1. Each summand in the second term can be written as $(\rho_{D_2} P_{\beta_2} \rho_{D_1}^{-1})\,\rho_{D_1}$. The first factor here is bounded by Lemma 2.3, and the second factor has the desired local smoothness by hypothesis (i) of Proposition 2.1.

Thus, all the hypotheses of Lemma 2.2 are verified and the limits (ii)′ exist. The remaining statements of the lemma are verified in the same way as the corresponding statements in Lemma 2.4. Again we note that $\operatorname{Ran} \Omega_\alpha^\pm \subset P_{\text{a.c.}}\mathscr{H}$ and that the orthogonality of channels (Theorem XI.36 of [19]) shows that $\operatorname{Ran} \Omega_\alpha^\pm \subset (1-\sum_{\beta_2} P_\beta^\mp)\,\mathscr{H}$. ∎

LEMMA 2.6. *Assume the hypotheses of Proposition* 2.1. *Then the following strong limits exist*:

(i) $\Omega_0^\pm = \text{s-lim}_{t\to\mp\infty} e^{itH} e^{-itH_0}$,

(ii) $W_0^\pm = \text{s-lim}_{t\to\pm\infty} e^{itH_0} e^{-itH}(1-\sum_{\alpha_3} P_{\alpha_3}^\pm)(1-\sum_{\beta_2} P_{\beta_2}^\pm)\,P_{\text{a.c.}}$.

*The adjoint of* $\Omega_0^\mp$ *is* $W_0^\pm$ *and the orthogonal projection onto the range of* $\Omega_0^\mp$ *is* $P_0^\pm = (1-\sum_{\alpha_3} P_\alpha^\pm)(1-\sum_{\beta_2} P_\beta^\pm)\,P_{\text{a.c.}}$.

*Proof.* The existence of the limits in (i) is standard [19, 23].

To show the existence of the limits in (ii) we first note that for $\psi \in \mathscr{H}$,

$$\left\| e^{-itH} P_0^\pm \psi - \left(1-\sum_{\alpha_3} P_{\alpha_3}\right)\left(1-\sum_{\beta_2} P_{\beta_2}\right) e^{-itH} P_{\text{a.c.}}\,\psi \right\|$$

tends to zero as $t \to \pm\infty$. Thus, the existence of the limits in (ii) is equivalent to the existence of the limits

$$\text{(ii)}' \quad W_0^\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_0}\left(1-\sum_{\alpha_3} P_{\alpha_3}\right)\left(1-\sum_{\beta_2} P_{\beta_2}\right) e^{-itH} P_{\text{a.c.}}.$$

To prove existence of these limits we need only verify the hypotheses of Lemma 2.2. Thus, we consider

$$\begin{aligned} H_0&\left(1-\sum_{\alpha_3} P_{\alpha_3}\right)\left(1-\sum_{\beta_2} P_{\beta_2}\right) - \left(1-\sum_{\alpha_3} P_{\alpha_3}\right)\left(1-\sum_{\beta_2} P_\beta\right) H \\ &= \left[H, \left(1-\sum_{\alpha_3} P_{\alpha_3}\right)\left(1-\sum_{\beta_2} P_{\beta_2}\right)\right] \\ &\quad - \sum_{i<j} V_{ij}\left(1-\sum_{\alpha_3} P_{\alpha_3}\right)\left(1-\sum_{\beta_2} P_{\beta_2}\right). \end{aligned}$$

This may be written as a sum of three terms.

$$\sum_{\alpha_3} [P_{\alpha_3}, H]\left(1 - \sum_{\beta_2} P_{\beta_2}\right), \tag{2.4}$$

$$\left(1 - \sum_{\alpha_3} P_\alpha\right) \sum_{\beta_2} [P_{\beta_2}, H] \tag{2.5}$$

and

$$-\sum_{i<j} V_{ij}\left(1 - \sum_{\alpha_3} P_{\alpha_3}\right)\left(1 - \sum_{\beta_2} P_{\beta_2}\right). \tag{2.6}$$

Since $[P_\alpha, H] = [P_\alpha, I_\alpha]$, terms which contribute to (2.4) have the factorization required in Lemma 2.2 by the proof used to control the operator (2.3) in the proof of Lemma 2.5. Similarly, the methods used to control the operator (2.2) show that the terms which contribute to (2.5) also factor as required by Lemma 2.2.

The operator (2.6) has two types of terms:

$$-V_{ij}(1 - P_{ij})\left(1 - \sum_{\beta_2} P_{\beta_2}\right), \tag{2.7}$$

where $P_{ij}$ is the sum of all $P_\alpha$ with $D(\alpha) = \{\{i, j\}, \{k\}, \{l\}\}$ and

$$V_{ij} P_\alpha \left(1 - \sum_{\beta_2} P_{\beta_2}\right), \tag{2.8}$$

where $\alpha$ is a three cluster channel which is not equal to $\{\{i, j\}, \{k\}, \{l\}\}$.

Terms of the form (2.8) have already been studied since they occur as terms which contribute to (2.4). Terms of the form (2.7) can be written as

$$V_{ij}(1 - P_{ij})\left(1 - \sum_{\beta_2, (ij) \subset D(\beta_2)} P_{\beta_2}\right)$$
$$- V_{ij}(1 - P_{ij}) \sum_{\beta_2, (ij) \not\subset D(\beta_2)} P_{\beta_2}.$$

The first of these terms has the required factorization since $|V_{ij}|^{1/2}$ is $H_0$-smooth [20], $|V_{ij}|^{1/2}\rho_{ij}^{-1}$ is bounded and $\rho_{ij}(1 - P_{ij})(1 - \sum_{(ij) \subset D_2} P_{D_2})$ is locally $H$-smooth on compact subintervals of $\sigma_{\text{a.c.}}(H)\backslash S_{\text{ex}}$ by hypothesis (iii) of Proposition 2.1. The second also has the required factorization since $|V_{ij}|^{1/2}$ is $H_0$ smooth, $(|V_{ij}|^{1/2}(1 - P_{ij})\,\rho_{ij}^{-1})(\rho_{ij} \sum_{(ij) \not\subset D_2} P_{D_2}\rho_{D_1}^{-1})$ is bounded (use Lemma 2.3), and $\rho_{D_1}$ is locally $H$-smooth by hypothesis (i) of Proposition 2.1. Thus, the limits (ii)′ exist.

Since $\operatorname{Ran} \Omega_0^{\mp}$ is orthogonal to $\operatorname{Ran} P_\alpha^{\pm}$ for all two cluster and three cluster channels $\alpha$ and $\Omega_0^{\mp}$ is a partial isometry, the final statements of the lemma follow by simple calculations, as was the case in Lemma 2.4. ∎

*Proof of Proposition* 2.1. Lemmas 2.4 and 2.5 identify the ranges of the channel wave operators as $\operatorname{Ran} \Omega_\alpha^{\mp} = \operatorname{Ran} P_\alpha^{\pm}$ if $\alpha$ is a two or three cluster channel. Lemma 2.6 identifies $\operatorname{Ran} \Omega_0^{\mp}$ as $\operatorname{Ran} P_0^{\pm}$, where $P_0^{\pm} = (1 - \sum_{\alpha_3} P_{\alpha_3}^{\pm})(1 - \sum_{\beta_2} P_{\beta_2}^{\pm})$. The orthogonality of channels [19] shows that $P_0^{\pm}$ is the orthogonal projection onto the complement of $\bigoplus_{\alpha \neq 0} \operatorname{Ran} P_\alpha^{\pm}$. Thus, $\bigoplus_\alpha \operatorname{Ran} P_\alpha^{\pm} = \mathscr{H}_{\text{a.c.}}$ and asymptotic completeness holds. ∎

## 3. A Technically More Useful Version of Proposition 2.1

In this section we restate Proposition 2.1 in a way which is technically more advantageous. Specifically, we replace the local smoothness hypotheses on operators (ii) and (iii) by local smoothness hypotheses on two other operators. The two statements, Propositions 2.1 and 3.1, are equivalent.

We use the notation $D_k$ to denote any cluster decomposition with precisely $k$ clusters. We write $D_k \subset D_l$ if $k \geqslant l$ and each cluster of $D_k$ is a subset of a cluster of $D_l$. We let $Q_2 = (1 - \sum_{D_2} P_{D_2})$ and $Q_3 = (1 - \sum_{D_3} P_{D_3})$. With this notation we can state the main result of this section.

Proposition 3.1. *Let $H$ satisfy the hypotheses of Theorem* 1.1, *and assume the potentials $V_{ij}$ are bounded. Suppose there exists a closed set $S_{\text{ex}}$ of measure zero such that all operators of the following types are locally $H$-smooth on compact subintervals of $\sigma_{\text{a.c.}}(H) \backslash S_{\text{ex}}$:*

(i)′ $\rho_{D_1}$,

(ii)′ $\rho_{D_2}(1 - P_{D_2})(1 - \sum_{D_3 \not\subset D_2} P_{D_3})\, Q_2$,

(iii)′ $\rho_{D_3}(1 - P_{D_3})\, Q_3 Q_2$.

*Then asymptotic completeness holds.*

*Remark* 1. The local smoothness of operators (i), (ii), and (iii) of Proposition 2.1 is equivalent to the local smoothness of (i)′, (ii)′, and (iii)′, although we will only show the implication in one direction. Proposition 3.1 follows obviously from Proposition 2.1 if the local Kato-smoothness of (i)′, (ii)′ and (iii)′ implies that of (i), (ii) and (iii).

2. Suppose $D$ and $D'$ are cluster decompositions and that $D''$ is the cluster decomposition with the greatest number of clusters which is refined by both $D$ and $D'$. Then $\rho_{D''}^{-1} P_D P_{D'} \rho_{D''}^{-1}$ and $\rho_{D''}^{-1} \rho_D \rho_{D'}$ are bounded. The

proofs of these facts are similar to the proof of Proposition 2.3. We will use the facts heavily in the proof of Lemma 3.1.

3. It is technically more convenient to deal with (ii)′ and (iii)′ than (ii) and (iii) (Note (i)′ coincides with (i).). The reasons for this will be clear in Section 4.

*Proof of Proposition* 3.1. It is clearly sufficient to prove that the local smoothness of (i)′, (ii)′, and (iii)′ imply that of (i), (ii), and (iii). Since (i) and (i)′ are identical, we need only consider (ii) and (iii).

The operator in (ii) can be written as

$$\begin{aligned}\rho_{D_2}(1-P_{D_2}) &= \rho_{D_2}(1-P_{D_2})\Big(1-\sum_{D_2'} P_{D_2'}\Big)+\rho_{D_2}(1-P_{D_2})\sum_{D_2'} P_{D_2'}\\ &= \rho_{D_2}(1-P_{D_2})\Big(1-\sum_{D_3\not\subset D_2} P_{D_3}\Big)Q_2\\ &\quad+\rho_{D_2}(1-P_{D_2})\sum_{D_3\not\subset D_2} P_{D_3}Q_2\\ &\quad+\rho_{D_2}(1-P_{D_2})\sum_{D_2'} P_{D_2'}. \end{aligned}\tag{3.1}$$

The first term in the last expression is locally $H$-smooth by (ii)′. The second term can be rewritten as

$$(\rho_{D_2}(1-P_{D_2})\,\rho_{D_2}^{-1})\Big(\sum_{D_3\not\subset D_2}\rho_{D_2}P_{D_3}\rho_{D_1}^{-1}\Big)(\rho_{D_1}Q_2\rho_{D_1}^{-1})\,\rho_{D_1}.$$

Using the fall off of subsystem eigenfunctions [3, 4, 17, 20] and Remark 2 (above), we see that the first three factors here are bounded. The last factor has the desired local smoothness by (i)′. Since $(1-P_{D_2})\,P_{D_2}=0$, the last term of (3.1) is equal to $\rho_{D_2}(1-P_{D_2})\sum_{D_2'\neq D_2}P_{D_2'}$. We rewrite this as

$$(\rho_{D_2}(1-P_{D_2})\,\rho_{D_2}^{-1})\Big(\sum_{D_2'\neq D_2}\rho_{D_2}P_{D_2'}\rho_{D_1}^{-1}\Big)\rho_{D_1}.$$

Again using the fall of subsystem eigenfunctions and Remark 2 (above), we see that the first two factors are bounded. The third factor is locally $H$-smooth by (i)′. Thus, the local smoothness of (i)′ and (ii)′ imply local smoothness of (ii).

The operator in (iii) can be rewritten as

$$\begin{aligned}&\rho_{D_3}(1-P_{D_3})\Big(1-\sum_{D_2\supset D_3}P_{D_2}\Big)\\ &\quad=\rho_{D_3}(1-P_{D_3})\,Q_2+\rho_{D_3}(1-P_{D_3})\sum_{D_2\not\supset D_3}P_{D_2}\\ &\quad=\mathrm{I}+\mathrm{II}+\mathrm{III},\end{aligned}$$

where

$$\mathrm{I} = \rho_{D_3}(1 - P_{D_3})\, Q_3 Q_2,$$

$$\mathrm{II} = \rho_{D_3}(1 - P_{D_3}) \sum_{D_3' \neq D_3} P_{D_3'} Q_2,$$

$$\mathrm{III} = \rho_{D_3}(1 - P_{D_3}) \sum_{D_2 \not\supset D_3} P_{D_2}.$$

The operator I has the desired local smoothness by (iii)′. The operator III can be rewritten as

$$(\rho_{D_3}(1 - P_{D_3})\, \rho_{D_3}^{-1}) \left( \sum_{D_2 \not\supset D_3} \rho_{D_3} P_{D_2} \rho_{D_1}^{-1} \right) \rho_{D_1}.$$

Remark 2 (above) and the fall off of subsystem eigenfunctions show that the first two factors here are bounded. The third is locally $H$-smooth by (i)′. Since II can be written as

$$(\rho_{D_3}(1 - P_{D_3})\, \rho_{D_3}^{-1}) \left( \rho_{D_3} \sum_{D_3' \neq D_3} P_{D_3'} Q_2 \right), \qquad \text{and} \qquad \rho_{D_3}(1 - P_{D_3})\, \rho_{D_3}^{-1}$$

is bounded, the local smoothness of II is implied by the local smoothness of operators of the form $\rho_{D_3} P_{D_3'} Q_2$ with $D_3 \neq D_3'$. Since $\rho_{D_3} P_{D_3'} \rho_{D_2}^{-1}$ is bounded for a suitably chosen $D_2$ (see Remark 2), the local smoothness of $\rho_{D_3} P_{D_3'} Q_2$ is implied by that of $\rho_{D_2} Q_2$. We rewrite $\rho_{D_2} Q_2$ as $\rho_{D_2}(1 - P_{D_2}) + \sum_{D_2' \neq D_2} \rho_{D_2} P_{D_2'}$. The first term here is locally smooth since we have already shown operators of type (ii) are locally $H$-smooth. The remaining term is locally $H$-smooth since it may be written as a sum of terms of the form $(\rho_{D_2} P_{D_2'} \rho_{D_1}^{-1})\, \rho_{D_1}$. The first factor is bounded by Remark 2. The second is locally $H$-smooth by (i)′. Thus, operators of type (iii) are locally $H$-smooth and Proposition 3.1 follows from Proposition 2.1. ∎

## 4. Reduction of Smoothness Estimates to Estimates Involving Subsystems

In this section we further reduce the problem of asymptotic completeness. In particular, we show that the operators (i)′, (ii)′, and (iii)′ in Proposition 3.1 have the required local smoothness properties if certain operators involving subsystem resolvents are well behaved.

DEFINITION. Suppose $A(z)\colon \mathscr{H} \to \mathscr{H}$ is an operator valued function which satisfies the following properties:

(a) $A(z)$ is analytic for $z \in \mathbb{C} \backslash \sigma_{\text{ess}}(H)$.

(b) $A(z)$ is uniformly bounded for $\operatorname{Re} z \leqslant C$, $|\operatorname{Im} z| \leqslant 1$.

(c) $\|A(z)\| \to 0$ as $\operatorname{Re} z \to -\infty$.

(d) $A(z)$ has norm continuous extensions $A(x \pm i0)$ to $\sigma_{\text{ess}}(H)$ from above and below.

Then $A(z)$ is called *norm well behaved on* $\mathbb{C} \backslash \sigma_{\text{ess}}(H)$. If (a), (b), and (c) hold, and if $A(z)$ has strongly continuous extensions $A(x \pm i0)$ to $\sigma_{\text{ess}}(H)$ from above and below, then $A(z)$ is called *strongly well behaved on* $\mathbb{C} \backslash \sigma_{\text{ess}}(H)$.

The principal result of this section is:

PROPOSITION 4.1. *Assume the hypotheses of Theorem* 1.1, *and assume the potentials* $V_{ij}$ *are bounded. The remaining hypotheses of Proposition* 3.1 *follow from the following statements*:

(a) $\rho_D(1-P_D)(z-H_D)^{-1}\rho_D$ *is norm well behaved for all* $D \neq D_1$.

(b) $\rho_{D_2}(1-P_{D_2})(z-H_{D_2})^{-1}\rho_{D_3}$ *is well behaved for all choices of* $D_3$ *and* $D_2$ *with* $D_3 \not\subset D_2$.

(c) $\rho_{D_3}(1-P_{D_3})(z-H_{D_3})^{-1}\rho_{D_3'}$ *is strongly well behaved for all choices of* $D_3$ *and* $D_3'$ *with* $D_3 \neq D_3'$, *and is norm well behaved unless* $D_3 = \{\{i,j\},\{k\},\{l\}\}$ and $D_3' = \{\{i\},\{j\},\{k,l\}\}$.

(d) $\rho_{D_3}(1-P_{D_3})(z-H_{D_3})^{-1}V_{D_3'}(1-P_{D_3'})(z-H_{D_3'})^{-1}\rho_{D_3''}$ *is norm well behaved if* $D_3 \neq D_3'$ and $D_3' \neq D_3''$.

(e) *There exists a closed set* $S_{\text{ex}} \subset \mathbb{R}$ *of measure zero such that* $\rho_{D_1}(z-H)^{-1}\rho_{D_1}$ *originally defined for* $z \in \mathbb{C} \backslash \sigma(H)$ *has norm continuous boundary values on* $\sigma_{\text{ess}}(H) \backslash S_{\text{ex}}$.

*Remarks.* 1. Theorems 8.1 of [18] shows that condition (e) follows from the hypotheses of Theorem 1.1. Thus, condition (e) is redundant in the proposition. In Sections 5 and 6 we will prove that conditions (a)–(d) also follow from the hypotheses of Theorem 1.1.

2. Condition (b) implies that $\rho_{D_2}(1-P_{D_2})(z-H_{D_2})^{-1}\rho_{D_2'}$ is norm well behaved for any $D_2' \neq D_2$. This is due to the fact that given $D_2' \neq D_2$, there exists $D_3 \subset D_2$ with $D_3 \subset D_2'$. For this $D_3$, the operator $\rho_{D_3}^{-1}\rho_{D_2'}$ is bounded. Similarly, either (a) or (b) implies that $\rho_{D_2}(1-P_{D_2})(z-H_{D_2})^{-1}\rho_{D_1}$ is norm well behaved. Condition (c) implies that

$$\rho_{D_3}(1-P_{D_3})(z-H_{D_3})^{-1}\rho_{D_2} \qquad \text{and} \qquad \rho_{D_3}(1-P_{D_3})(z-H_{D_3})^{-1}\rho_{D_1}$$

are well behaved for any choice of $D_2$. In what follows we will freely use these implications without comment.

We now outline the proof of Proposition 4.1. For the sake of brevity, the details of many messy algebraic calculations have been omitted.

To prove Proposition 4.2, we first note that condition (e) implies te required local smoothness of $\rho_{D_1}$(see Theorem XIII.30 of [20]). This takes care of operators of type (i)′ in Proposition 3.1. To deal with operators of types (ii)′ and (iii)′ let $A(z)$ denote the 13 component vector, whose first seven components are the operators of the form

$$\rho_{D_2}(1-P_{D_2})\left(1-\sum_{D_3\subset D_3} P_{D_3}\right)Q_2(z-H)^{-1},$$

and whose six remaining components are the operators of the form $\rho_{D_3}(1-P_{D_3})\,Q_3Q_2(z-H)^{-1}$. It follows from Theorem XIII-25 of [20] that to prove Proposition 4.1 we need only show the hypotheses imply that that for $\operatorname{Re} z$ in any compact subinterval $K\subset\sigma_{\mathrm{ess}}(H)\backslash S_{\mathrm{ex}}$ we have

$$\sup_{\substack{0<|\operatorname{Im} z|\leq 1\\ \operatorname{Re} z\in K}} \sqrt{\operatorname{Im} z}\;\|A(z)\|<\infty. \tag{4.1}$$

(Here $\|A(z)\|$ denotes the sum of the norms of the entries of $A(z)$.)

To prove (4.1) we first show that for $\operatorname{Im} z\neq 0$, $A(z)$ satisfies an equation of the form

$$A(z)=B(z)+C(z)\,A(z). \tag{4.2}$$

The hypotheses of Proposition 4.1 are then used to show that $C(z)$ is analytic for $z\in\mathbb{C}\backslash\sigma_{\mathrm{ess}}(H)$ with strongly continuous boundary values $C(x\pm i0)$ as $z$ approaches $\sigma_{\mathrm{ess}}(H)$ from above or below. The hypotheses of Proposition 4.1 show that $B(z)$ satisfies an estimate of the form

$$\sup_{\substack{0<|\operatorname{Im} z|\leq 1\\ \operatorname{Re} z\in K}} \sqrt{\operatorname{Im} z}\;\|B(z)\|<\infty,$$

for any compact subinterval $K\subset\sigma_{\mathrm{ess}}(H)\backslash S_{\mathrm{ex}}$. The result (4.1) would follow if we could show $(1-C(z))^{-1}$ existed and were uniformly bounded for $\operatorname{Re} z\in K$ for all compact subintervals $K\subset\sigma_{\mathrm{ess}}(H)\backslash S_{\mathrm{ex}}$.

We have not been able to show the invertibility of $(1-C(z))$, so some more most be done to prove (4.1). We iterate (4.2) once to obtain

$$A(z)=B(z)+C(z)\,B(z)+C(z)^2A(z).$$

Then by splitting up and regrouping terms in this equation in a way which depends on two parameters $R_1$ and $R_2$, we obtain new equations of the form

$$A(z) = F_{R_1,R_2}(z) + G_{R_1,R_2}(z)\, A(z). \tag{4.3}$$

We then show that for any choice of $R_1$ and $R_2$ we have

$$\sup_{\substack{0 < |\mathrm{Im}\, z| \leqslant 1 \\ \mathrm{Re}\, z \in K}} \sqrt{\mathrm{Im}\, z}\, \|F_{R_1,R_2}(z)\| < \infty,$$

for any compact subinterval $K \subset \sigma_{\mathrm{ess}}(H) \backslash S_{\mathrm{ex}}$. Furthermore, for appropriate $R_1$, $R_2$, $G_{R_1,R_2}(z)$ has norm well behaved entries and $(1 - G_{R_1,R_2}(z))^{-1}$ exists and is uniformly bounded for Re $z$ restricted to any compact set. The result (4.1) follows, and consequently, Proposition 4.1 is proved.

The remainder of this section will be devoted to a description of how one obtains the equations (4.2) and (4.3), and how one proves that they have all the properties claimed above.

We now derive the equations (4.2). Consider any one of the first seven entries of $A(z)$. By a resolvent identity we have

$$\begin{aligned}
&\rho_{D_2}(1 - P_{D_2})\left(1 - \sum_{D_3 \not\subset D_2} P_{D_3}\right) Q_2 (z - H)^{-1} \\
&\quad = \rho_{D_2}(1 - P_{D_2})(z - H_{D_2})^{-1}\left(1 - \sum_{D_3 \not\subset D_2} P_{D_3}\right) Q_2 \\
&\qquad + \rho_{D_2}(1 - P_{D_2})(z - H_{D_2})^{-1}\left[\left(1 - \sum_{D_3 \not\subset D_2} P_{D_3}\right) Q_2 H \right. \\
&\qquad \left. - H_{D_2}\left(1 - \sum_{D_3 \not\subset D_2} P_{D_3}\right) Q_2\right](z - H)^{-1}.
\end{aligned} \tag{4.4}$$

Hypothesis (a) of Proposition 4.1 and Theorems XIII.25 and XIII.30 of [20] show that the first term $X(z)$ on the right-hand side satisfies an estimate of the form

$$\sup_{\substack{0 < |\mathrm{Im}\, z| \leqslant 1 \\ \mathrm{Re}\, z \in K}} \sqrt{\mathrm{Im}\, z}\, \|X(z)\| < \infty \tag{4.5}$$

for any compact interval $K$. The factor in the square brackets in the second term of (4.4) can be written as

$$\sum_{D_3 \not\subset D_2} \mathrm{V}_{D_3}\left(1 - \sum_{D_3' \not\subset D_2} P_{D_3'}\right) Q_2 + \left[\left(1 - \sum_{D_3 \not\subset D_2} P_{D_3}\right), H\right] Q_2$$

$$+\left(1-\sum_{D_3\not\subset D_2} P_{D_3}\right)[Q_2, H]$$

$$=\sum_{D_3\not\subset D_2} \rho_{D_3}(\rho_{D_3}^{-1}\mathrm{V}_{D_3}\rho_{D_3}^{-1}\Big)\rho_{D_3}\left(1-\sum_{D_3'\not\subset D_2} P_{D_3'}\right)Q_2$$

$$+\sum_{D_3\not\subset D_2}[H, P_{D_3}]\,Q_2+\left(1-\sum_{D_3\not\subset D_2} P_{D_3}\right)\sum_{D_2'}[H, P_{D_2'}]$$

$$=\sum_{D_3\not\subset D_2} \rho_{D_3}(\rho_{D_3}^{-1}\mathrm{V}_{D_3}\rho_{D_3}^{-1})\,\rho_{D_3}\left(1-\sum_{D_3'\not\subset D_2} P_{D_3'}\right)Q_2$$

$$+\sum_{D_3\not\subset D_3}[\mathrm{I}_{D_3}, P_{D_3}]\,Q_2+\left(1-\sum_{D_3\not\subset D_2} P_{D_3}\right)\sum_{D_3'}[\mathrm{I}_{D_2}, P_{D_2}],$$

where $\mathrm{I}_D=\mathrm{V}-\mathrm{V}_D$. Let the three terms which occur in the final expression here be I, II, and III, respectively. Lemma 2.3 shows that III may be written as $\rho_{D_1}(\rho_{D_1}^{-1}\,\mathrm{III}\,\rho_{D_1}^{-1})\,\rho_{D_1}$, where the middle factor is bounded. Thus, Remark 2 (above) and hypotheses (a) and (e) show that III gives rise to a term in (4.4) which satisfies an estimate of the form (4.5) if $K$ is any compact subinterval of $\sigma_{\mathrm{ess}}(H)\backslash S_{\mathrm{ex}}$. Next, Lemma 2.3 shows that $[\mathrm{I}_{D_3}, P_{D_3}]$ is a sum of terms of the form $\rho_{D_2'}$ (bounded operator) $\rho_{D_2'}$ for various choices of $D_2'$. Thus, II is a sum of terms of the form $\rho_{D_2'}$ (bounded operator) $\rho_{D_2'}Q_2$. In each such term we write $\rho_{D_2'}Q_2$ as

$$\begin{aligned}&\rho_{D_2'}(1-P_{D_2'})\left(1-\sum_{D_3\not\subset D_2'} P_{D_3}\right)Q_2\\&\quad+\rho_{D_2'}(1-P_{D_2'})\sum_{D_3\not\subset D_2'} P_{D_3}Q_2\\&\quad+\rho_{D_2'}(1-P_{D_2'})\sum_{D_2''\neq D_2'} P_{D_2''}\\&\quad+\rho_{D_2'}\sum_{D_2''\neq D_2} P_{D_2''}.\end{aligned}\tag{4.6}$$

All but the first of these terms have the form $\rho_{D_1}$ (bounded operator) $\rho_{D_1}$ by Lemma 2.3. Their contributions in (4.4) satisfy estimates of the type (4.5) for compact subintervals $K\subset\sigma_{\mathrm{ess}}(H)\backslash S_{\mathrm{ex}}$ for the same reasons that III gave rise to such terms. The first term in (4.6) is a term of type (ii)′ in Proposition 3.1. In (4.4) it gives rise to a term which is a factor which is norm well behaved by hypothesis (b) and Remark 2,

$$\rho_{D_2}(1-P_{D_2})(z-H_{D_2})^{-1}\rho_{D_2'}\quad\text{(bounded operator)},$$

times one of the first seven entries of $A(z)$.

We now must deal with the term I. Using $(1-P_{D_3})^2=(1-P_{D_3})$, we have

$$\begin{aligned}\rho_{D_3}\Big(1-\sum_{D_3'\not\subset D_2} P_{D_3'}\Big)Q_2 &= \rho_{D_3}Q_3Q_2+\rho_{D_3}\sum_{D_3'\subset D_2} P_{D_3'}\\ &= \rho_{D_3}(1-P_{D_3})\,Q_3Q_2+\rho_{D_3}(1-P_{D_3})\sum_{D_3'\neq D_3} P_{D_3'}Q_2\\ &\quad -\rho_{D_3}\sum_{D_3'\neq D_3} P_{D_3'}Q_2+\rho_{D_3}\sum_{D_3'\subset D_2} P_{D_3'}Q_2. \end{aligned} \tag{4.7}$$

The first of the terms in the right hand side of (4.7) is an operator of type (iii)′ in Proposition 3.1. It gives rise to a term in I which gives rise to a term in (4.4) which equals a norm well behaved factor,

$$\rho_{D_2}(1-P_{D_2})(z-H_{D_2})^{-1}\rho_{D_3} \quad \text{(bounded operator)}$$

(with $D_3\not\subset D_2$), times one of the final six entries of $A(z)$. In the remaining terms on the right-hand side of (4.7), every $D_3'$ which occurs satisfies $D_3'\neq D_3$. Thus, Lemma 2.3 shows that these terms are sums of operators of the form

$$\text{(bounded operator)}\ [(1+|x_{D_3'}|^2)^{-1}\rho_{D_3}P_{D_3'}\rho_{D_2''}^{-1}]\,\rho_{D_2''}Q_2,$$

where the factor in the square brackets is bounded (here $D_2''$ is chosen with $D_3\subset D_2''$ and $D_3'\subset D_2''$). The factor $\rho_{D_2''}Q_2$ has the same form as the operator (4.6). It gives rise to the same types of terms in (4.4) which are generated by II, because the $(1+|x_{D_3'}|^2)^{-1}$ factor can be commuted all the way to the left in the term I, and $\rho_{D_2''}^{-1}(1+|x_{D_3}|^2)^{-1}\rho_{D_3}$ is bounded.

Thus, we see that the first seven entries of $A(z)$ equal sums of terms satisfying estimates of the type (4.5) for compact intervals $K\subset\sigma_{\text{ess}}(H)\backslash S_{\text{ex}}$ plus norm well behaved operators times entries of $A(z)$. Consequently, we have the first seven of the thirteen coupled equations which make up the system (4.2).

Next consider the final six entries of $A(z)$. By a resolvent identity we have

$$\begin{aligned}&\rho_{D_3}(1-P_{D_3})\,Q_3Q_2(z-H)^{-1}\\ &\quad=\rho_{D_3}(1-P_{D_3})(z-H_{D_3})^{-1}Q_3Q_2\\ &\qquad+\rho_{D_3}(1-P_{D_3})(z-H_{D_3})^{-1}[Q_3Q_2H-H_{D_3}Q_3Q_2](z-H)^{-1}. \end{aligned}\tag{4.8}$$

Theorems XIII.25 and XIII.30 of [20], hypothesis (a), and Remark 2 (above) show that the first term on the right-hand side of (4.8) satisfies an estimate of the type (4.5) for any compact interval $K$. The factor in the square brackets in the second term on the right-hand side of (4.8) may be written as

$$\begin{aligned} &\mathrm{I}_{D_3} Q_3 Q_2 + [Q_3, H]\, Q_2 + Q_3[Q_2, H] \\ &\quad = \mathrm{I}_{D_3} Q_3 Q_2 + \sum_{D_3'} [H, P_{D_3'}]\, Q_2 + \sum_{D_2} Q_3[H, P_{D_2}] \\ &\quad = \mathrm{I}_{D_3} Q_3 Q_2 + \sum_{D_3'} [I_{D_3'}, P_{D_3'}]\, Q_2 + \sum_{D_2} Q_3[\mathrm{I}_{D_2}, P_{D_2}], \end{aligned}$$

where $\mathrm{I}_D = \mathrm{V} - \mathrm{V}_D$. Let the three terms which occur here be denoted by IV, V, and VI, respectively. By the same analysis used to study III, we see that VI gives rise to a term in (4.8) which satisfies an estimate of the form (4.5) for compact subintervals $K \subset \sigma_{\mathrm{ess}}(H) \backslash S_{\mathrm{ex}}$. The term V is dealt with the same way as the term II was, except that hypothesis (c) is used in place of hypothesis (b) to show that

$$\rho_{D_3}(1 - P_{D_3})(z - H_{D_3})^{-1} \rho_{D_2'} \quad \text{(bounded operator)}$$

is norm well behaved. Similarly, the term IV is handled in much the same way that the term I was handled. Again, hypothesis (c) must be used in place of hypothesis (b), however, here some factors which occur are only strongly well behaved.

We now have constructed the equations (4.2). We wish to view these equations as two coupled equations for the two components of $A(z)$ (The first seven entries of $A(z)$ form the first component $A^{(1)}(z)$; the remaining six entries make up the second component $A^{(2)}(z)$). Explicitly, we have

$$\begin{bmatrix} A^{(1)}(z) \\ A^{(2)}(z) \end{bmatrix} = \begin{bmatrix} B^{(1)}(z) \\ B^{(1)}(z) \end{bmatrix} + \begin{bmatrix} C^{(1,1)}(z) & C^{(1,2)}(z) \\ C^{(2,1)}(z) & C^{(2,2)}(z) \end{bmatrix} \begin{bmatrix} A^{(1)}(z) \\ A^{(2)}(z) \end{bmatrix}$$

The operator $B(z)$ satisfies the estimate (4.3), and the operators $C^{(1,1)}(z)$, $C^{(1,2)}(z)$, and $C^{(2,1)}(z)$ are norm well behaved. The operator $C^{(2,2)}(z)$ contain entries which are either norm well behaved or strongly well behaved. By employing the methods of Section XIII.5 of [20] it is easy to see that the entries of $C^{(1,1)}(z)$ and $C^{(1,2)}(z)$ are compact. By some simple functional analysis (see Lemma V.24 of [10]), the product of a compact norm well behaved operator and a strongly well behaved operator is norm well behaved and compact. Consequently, hypothesis (d) of Proposition 4.1 and explicit computation show that the square of $C(z)$ has norm well behaved entries.

We now let $F_{0,0}(z) = B(z) + C(z)\, B(z)$ and $G_{0,0}(z) = [C(z)]^2$. Then by iterating (4.2) once we obtain the special case of (4.3) which has $R_1 = R_2 = 0$. Viewing $G_{0,0}(z)$ as a two by two matrix as was done with

$C(z)$, let $X(z)$ denote any non-compact operator in the 2, 2 entry of $G_{0,0}(z)$. Such an operator has the form

$$X(z)=\rho_{D_3}(1-P_{D_3})(z-H_{D_3})^{-1}\mathrm{V}_{D_3'}(1-P_{D_3'})(z-H_{D_3'})^{-1}\mathrm{V}_{D_3''}\rho_{D_3''}^{-1}$$

with $D_3 \neq D_3' \neq D_3''$. Because of the non-compactness, there exists some $D_2$ with $D_3 \subset D_2$, $D_3' \subset D_2$, and $D_3'' \subset D_2$. Let $x_{D_2} \in \mathbb{R}^{2n}$ denote the clustered Jacobi coordinates within the clusters of $D_2$, and let $p \in \mathbb{R}^n$ denote the momentum conjugate to the vector between the centers of mass of the clusters. Then this non-compact operator is decomposable according to the direct integral decomposition

$$\mathscr{H}=\int_{\mathbb{R}^n}^{\oplus} L^2(\mathbb{R}^{2n}, dx_{D_2})\, dp$$

(see p. 281 of [20]). The fibers of the operator are compact on $L^2(\mathbb{R}^{2n}, dx_{D_2})$ by the methods of Section XIII.5 of [20], and as $|p| \to \infty$, the norm of the fiber at $p$ tends to zero. Consequently, we have

$$\lim_{R_1 \to \infty} \|X(z)\, \chi_{|x_{D_2}| > R_1}\| = 0 \tag{4.9}$$

uniformly for $z$ in any compact set.

We now imagine explicitly writing out equation (4.3) for the case $R_1 = R_2 = 0$, i.e.,

$$A(z) = F_{0,0}(z) + G_{0,0}(z)\, A(z). \tag{4.10}$$

In every term of this equation which contains a non-compact factor from the 2, 2 entry of $G_{0,0}(z)$ we insert a factor of $(\chi_{|x_{D_2}| > R_1} + \chi_{|x_{D_2}| \leqslant R_1})$ between that non-compact entry of $G_{0,0}(z)$ and the following entry of $A(z)$. Here $D_2$ is chosen as in the discussion above. If $R_1$ is large, the norm of the term containing $\chi_{|x_{D_2}| > R_1}$ is arbitrarily small, uniformly for $z$ in a compact set due to (4.9). The term containing

$$\chi_{|x_{D_2}| \leqslant R_1} = \chi_{|x_{D_2}| \leqslant R_1}\, \rho_{D_2}^{-1} \rho_{D_2}$$

can be factored so that the factor $\chi_{|x_{D_2}| \leqslant R_1} \rho_{D_2}^{-1}$ is associated with the $G_{0,0}(z)$ entry and the $\rho_{D_2}$ is associated with the entry of $A(z)$. This entry of $A(z)$ is one of the last six entries of $A(z)$, but by using the fall off in the $\rho_{D_2}$ factor, $\rho_{D_2} A_j(z)$ $(8 \leqslant j \leqslant 13)$ can be rewritten as a sum of bounded operators times terms of the form $A_j(z)$ for $1 \leqslant j \leqslant 7$ plus bounded operators times $\rho_{D_1}(z-H)^{-1}$. The algebra here is very similar to what was done in handling terms of type V in equation (4.8). The new terms which now appear as bounded operators times $\rho_{D_1}(z-H)^{-1}$ are moved in equation (4.10) so that they become part of the $F(z)$ term. The new terms which involve $A_j(z)$ for $1 \leqslant j \leqslant 7$ are similarly moved so that they are regarded as products of entries in the 2, 1 entry of $G(z)$ times $A_j(z)$'s for $1 \leqslant j \leqslant 7$.

The compact entries of the 2, 2 entry of $G_{0,0}(z)$ are dealt with in a similar fashion, except that for them we insert $(\chi_{|x| \leqslant R_1} + \chi_{|x| \leqslant R_1})$, where $|x| = |x_{D_1}|$ grows in all directions of configuration space. Then the terms containing

$$\chi_{|x| \leqslant R_1} = \chi_{|x| \leqslant R_1} \rho_{D_1}^{-1} \rho_{D_1}$$

can be moved into the $F$ term of equation of (4.10). The terms containing $\chi_{|x| > R_1}$ are arbitrarily small for large $R_1$ since $\chi_{|x| > R_1}$ goes strongly to zero and the product of a compact operator times an operator which goes strongly to zero goes to zero in norm.

After doing the above algebra, we arrive at a new system

$$A(z) = F_{R_1,0}(z) + G_{R_1,0}(z)\, A(z). \tag{4.11}$$

For large values of $R_1$, the 2, 2 entry of $G_{R_1,0}(z)$ can be made arbitrarily small for $z$ in any compact subset of the closed cut plane, cut along $\sigma_{\text{ess}}(H)$. The 2, 1 entry now contains some huge terms, as does $F_{R_1,0}(z)$ for $R_1$ large.

We now obtain equation (4.3) from equation (4.11). We imagine writing (4.11) out explicitly. We then insert $(\chi_{|x| > R_2} + \chi_{|x| \leqslant R_2})$ after each entry of the 1, 1 and 1, 2 entries in the equation. The terms containing $\chi_{|x| \leqslant R_2}$ are moved into the $F(z)$ term. The 1, 1 and 1, 2 entries of the $G_{R_1,0}(z)$ term are compact, so when they are multiplied by $\chi_{|x| > R_1}$, the product is small for large $R_1$. We thus arrive at equation (4.3). Note that due to remark 1 above we have

$$\sup_{\substack{0 < |\mathrm{Im}\, z| \leqslant 1 \\ \mathrm{Re}\, z \in K}} \sqrt{\mathrm{Im}\, z}\, \|F_{R_1,R_2}(z)\| < \infty \tag{4.12}$$

for any choice of $R_1$ and $R_2$ and any compact subinterval $K \subset \sigma_{\text{ess}}(H) \backslash S_{\text{ex}}$. Furthermore, the operator $G_{R_1,R_2}(z)$ when thought of as a two by two matrix has the form

$$G_{R_1,R_2}(z) = \begin{bmatrix} g_1(R_2, z) & g_2(R_2, z) \\ g_3(R_1, z) & g_4(R_1, z) \end{bmatrix} \tag{4.13}$$

where

$$\lim_{R_2 \to \infty} \|g_j(R_2, z)\| = 0 \qquad \text{for} \quad j = 1, 2,$$

$$\lim_{R_1 \to \infty} \|g_3(R_1, z)\| = \infty,$$

$$\lim_{R_1 \to \infty} \|g_4(R_1, z)\| = 0$$

with all limits uniform for $z$ in compact subsets $K$ the closed cut plane cut along $\sigma_{\text{ess}}(H)$. It is not hard to see that by first choosing $R_1$ very large (to make $g_4(R_1, z)$ very small) and then choosing $R_2$ very large (to make $g_1(R_2, z)$ and $g_2(R_2, z)$ very small), some large power of $G_{R_1,R_2}(z)$ has norm less than one for $z \in K$, where $K$ is a compact subset of the closed cut plane. Thus, for $z \in K$ and this choice of $R_1$ and $R_2$, $(1 - G_{R_1,R_2}(z))^{-1}$ exists since $(1-y)^{-1} = (1 + y + y^2 + \cdots + y^{n-1})(1-y^n)^{-1}$ for any $n$ if $|y|^n < 1$. Consequently, (4.1) follows by solving equation (4.3) and using (4.12). This completes our outline of the proof of Proposition 4.1, since the local smoothness results follow from (4.1) (see Theorem XIII.30 of [20]).

## 5. Subsystem Resolvents and Generic Threshold Behavior

The purpose of this section is to collect various facts about subsystem resolvents which are required for the verification of some of the hypotheses of Proposition 4.1. All of the results of this section would follow from Theorem 8.1 of [18] if the estimates in Theorem 8.1 of [18] did not blow up at thresholds. Since [18] does not distinguish between generic and non-generic threshold behavior, the estimates of [18] do blow up. So, in some sense, this section is devoted to improving the estimates of [18] near thresholds under the additional hypothesis that the threshold behavior is generic.

Let us first consider a two-body Schrödinger operator $h = h_0 + V = -\Delta/2m + V$ on $L^2(\mathbb{R}^n)$ with $n \geqslant 3$. Assume $V(x) = (1+x^2)^{-\gamma} Y(x)$, where $\gamma > 1$ and $Y \in L^\infty(\mathbb{R}^n)$. Choose $\eta \in (1, \gamma)$ and let $\rho(x) = (1+x^2)^{-\eta/2}$. Then by standard arguments (see, e.g., [19]) the operator valued function $\rho(z - h_0)^{-1} V \rho^{-1}$ is norm well behaved on $\mathbb{C} \setminus [0, \infty)$. In addition, this operator valued function is compact for all $z$ in the closed cut plane. If $h$ has no positive energy eigenvalues, then, by a theorem of Agmon [1, 20], the number 1 is not an eigenvalue of $\rho(z-h_0)^{-1} V \rho^{-1}$ for $z = x \pm i0$, $x \in (0, \infty)$. The assumption that $h$ has no threshold eigenvalue and no threshold resonances means that 1 is not an eigenvalue of $\rho(0 - h_0)^{-1} V \rho^{-1}$. Since the mapping $\lambda \to \rho(0-h_0)^{-1}(\lambda V)\,\rho^{-1}$ is linear in $\lambda$ with compact operator values, there is only a discrete set of $\lambda$'s for which $h(\lambda) = h_0 + \lambda V$ has a threshold eigenvalue or threshold resonance.

Lemma 5.1. *Let $N$ be 3 or 4, and assume the hypotheses of Theorem 1.1. Assume (for convenience) that the $V_{ij}$ are bounded. Then for each pair $ij$, $\rho_{ij}(z - H_{ij})^{-1}(1 - P_{ij})\,\rho_{ij}$ is norm well behaved on $\mathbb{C} \setminus [0, \infty)$.*

*Proof.* Let $\zeta_1 = x_{ij}$, $\zeta_2, \ldots, \zeta_{N-1}$ be Jacobi coordinates, and let $p_2, p_3, \ldots, p_{N-1}$ be the momenta conjugate to $\zeta_2, \zeta_3, \ldots, \zeta_{N-1}$, respectively.

The operator valued function $\rho_{ij}(z-H_{ij})^{-1}(1-P_{ij})\,\rho_{ij}$ is fibered with respect to the decomposition $\mathscr{H}=\int_{\mathbb{R}^{(N-2)n}}^{\oplus} L^2(\mathbb{R}^n)\,dp_2\,dp_3\cdots dp_{N-1}$, with fibers $\rho_{ij}(z-c_2p_2^2-c_3p_3^2-\cdots-c_{N-1}p_{N-1}^2-h_{ij})^{-1}(1-P_{ij})\,\rho_{ij}$, for some constants $c_2, c_3, ..., c_{N-1}$. Thus, it is sufficient to prove $\rho_{ij}(z-h_{ij})^{-1}(1-P_{ij})\,\rho_{ij}$ is norm well behaved as an $L^2(\mathbb{R}^n)$-operator valued function.

If $z_0 \in \mathbb{C}$ is not an element of $\sigma_{\text{ess}}(h_{ij}) = [0, \infty)$, then $\rho_{ij}(z-h_{ij})^{-1}(1-P_{ij})\,\rho_{ij}$ is clearly analytic near $z_0$. Furthermore,

$$\begin{aligned}
&\rho_{ij}(z-h_{ij})^{-1}(1-P_{ij})\,\rho_{ij}\\
&\quad=\rho_{ij}(z-h_0)^{-1}(1-P_{ij})\,\rho_{ij}\\
&\quad\quad+\rho_{ij}(z-h_0)^{-1}V_{ij}\rho_{ij}^{-1}(1-\rho_{ij}(z-h_0)^{-1}V\rho_{ij}^{-1})^{-1}\\
&\quad\quad\times\rho_{ij}(z-H_0)^{-1}(1-P_{ij})\,\rho_{ij}.
\end{aligned}\tag{5.1}$$

As mentioned above, standard arguments show that $\rho_{ij}(z-h_0)^{-1}V_{ij}\rho_{ij}^{-1}$ and $\rho_{ij}(z-h_0)^{-1}(1-P_{ij})\,\rho_{ij}$ are norm well behaved. Since $h_{ij}$ has no positive eigenvalues, no threshold eigenvalues, and no threshold resonances, the right-hand side of (5.1) has norm continuous boundary values as $z$ approaches $[0, \infty)$ from above or below. Furthermore if $\operatorname{Re} z$ is large and negative, one can expand the inverse term on the right-hand side of (5.1) by using geometric series. By doing so, one sees that the right-hand side of (5.1) tends to zero in norm as $\operatorname{Re} z \to -\infty$. The lemma follows. ∎

In the $N=3$ case [11] we assumed the three two-body subsystems had no threshold resonances and no threshold bound states. To do the $N=4$ case we must also assume the three-body subsystems have no threshold resonances and no threshold bound states. To make a precise statement of what this means we begin by deriving some formulas involving three body resolvents. In particular, let $h=h_0+V_{12}+V_{13}+V_{23}$ be a three-body Hamiltonian on $L^2(\mathbb{R}^{2n})$, $n\geqslant 3$, whose two-body subsystems have no threshold resonances and no threshold bound states. Let

$$A_0(z)=\rho_{123}(z-h)^{-1}(1-P_{123})\,\rho_{123}$$

and

$$A_{ij}(z)=\rho_{ij}(1-P_{ij})(1-P_{12}-P_{13}-P_{23})(z-h)^{-1}(1-P_{123})\,\rho_{123},$$

where $\rho_{ij}=(1+x_{ij}^2)^{-\eta/2}$ with $\eta$ as above, and $\rho_{123}=(1+x_{ij}^2+\zeta_{ij}^2)^{-\eta/2}$ with $(x_{ij}, \zeta_{ij})$ Jacobi coordinates. The operators $A_0(z)$ and $A_{ij}(z)$ satisfy a system of coupled equations whose kernel is compact. The equations are obtained as follows with $Q=(1-P_{12}-P_{13}-P_{23})$:

$$A_{ij}(z)=\rho_{ij}(1-P_{ij})(z-h_{ij})^{-1}Q(1-P_{123})\,\rho_{123}$$

$$
\begin{aligned}
&+\rho_{ij}(1-P_{ij})(z-h_{ij})^{-1}\{h_{ij}Q-Qh\}(z-h)^{-1}(1-P_{123})\,\rho_{123}\\
&=\rho_{ij}(1-P_{ij})(z-h_{ij})^{-1}Q(1-P_{123})\,\rho_{123}\\
&\quad+\rho_{ij}(1-P_{ij})(z-h_{ij})^{-1}\rho_{123}(\rho_{123}^{-1}[h,Q]\,\rho_{123}^{-1})\,A_0(z)\\
&\quad-\rho_{ij}(1-P_{ij})(z-h_{ij})^{-1}\rho_{ik}(\rho_{ik}^{-1}\mathrm{V}_{ik}\rho_{ik}^{-1})\,A_{ik}(z)\\
&\quad+\rho_{ij}(1-P_{ij})(z-h_{ij})^{-1}\rho_{ik}(\rho_{ik}^{-1}\mathrm{V}_{ik})(P_{ik}(P_{ij}+P_{jk})\,\rho_{123}^{-1})\,A_0(z)\\
&\quad-\rho_{ij}(1-P_{ij})(z-h_{ij})^{-1}\rho_{jk}(\rho_{jk}^{-1}\mathrm{V}_{jk}\rho_{jk}^{-1})\,A_{jk}(z)\\
&\quad+\rho_{ij}(1-P_{ij})(z-h_{ij})^{-1}\rho_{jk}(\rho_{jk}^{-1}\mathrm{V}_{jk})(P_{jk}(P_{ij}+P_{ik})\,\rho_{123}^{-1})\,A_0(z) \qquad (5.2)
\end{aligned}
$$

$$
\begin{aligned}
A_0(z)&=\rho_{123}(P_{12}+P_{13}+P_{23}+Q)(z-h)^{-1}(1-P_{123})\,r_{123}\\
&=\sum_{i<j}\rho_{123}P_{ij}(z-h_{ij})^{-1}(1-P_{123})\,\rho_{123}\\
&\quad+\sum_{i<j}\rho_{123}P_{ij}(z-h_{ij})^{-1}\rho_{123}(\rho_{123}^{-1}P_{ij}(\mathrm{V}_{ik}+\mathrm{V}_{jk})\,\rho_{123}^{-1})\,A_0(z)\\
&\quad+\rho_{123}(z-h_0)^{-1}Q(1-P_{123})\,\rho_{123}\\
&\quad+\rho_{123}(z-h_0)^{-1}\rho_{123}(\rho_{123}^{-1}[h,Q]\,\rho_{123}^{-1})\,A_0(z)\\
&\quad+\sum_{i<j}\rho_{123}(z-h_0)^{-1}\rho_{ij}(\rho_{ij}^{-1}\mathrm{V}_{ij}\rho_{ij}^{-1})\,A_{ij}(z)\\
&\quad-\sum_{i<j}\rho_{123}(z-h_0)^{-1}\rho_{ij}(\rho_{ij}^{-1}\mathrm{V}_{ij})(P_{ij}+P_{jk})\,\rho_{123}^{-1}\,A_0(z). \qquad (5.3)
\end{aligned}
$$

If we write equations (5.2) and (5.3) as a four component system $A(z)=B(z)+C(z)\,A(z)$, then it is not hard to check that $B(z)$ and $C(z)$ are norm well behaved on $\mathbb{C}\backslash\sigma_{\mathrm{ess}}(h)$ (see, e.g., Proposition 5.1 of [9] for details). Also, $C(z)$ is compact and tends to 0 in norm as $\operatorname{Re} z\to-\infty$. If $z_0$ is not an eigenvalue or threshold for $h$, then by Theorem 8.1 of [18], $A_0(z_0)$ is bounded. Thus, we can view the equations of (5.2) as a three component system

$$
\begin{aligned}
A'(z_0)&=B'(z_0)+C'(z_0)\,A'(z_0)\\
&=(B'(z_0)+C'(s_0)[\chi_{|x|<R}\rho_{123}^{-1}]\,DA_0(z_0)\\
&\quad+C'(z_0)\,\chi_{|x|>R}A'(z_0), \qquad (5.4)
\end{aligned}
$$

where $D$ is a bounded operator. The term inside the parentheses in (5.4) is bounded for each fixed $R$. The operator $C'(z_0)\,\chi_{|x|>R}$ tends to zero in norm as $R\to\infty$. Thus, one can solve equations (5.4) for large $R$ to see that the operators $A_{ij}(z_0)$ are bounded.

The assumption that $h$ has no threshold bound states and no threshold eigenvalues means that the number 1 is not on eigenvalue of $C(z)$ when $z$ is a threshold.

If $h$ has no eigenvalues embedded in $\sigma_{\text{ess}}(h)$, has no threshold eigenvalues, and has no threshold resonances, then we can conclude from the above discussion that $A_0(z)$ and $A_{ij}(z)$ are norm well behaved except possibly for poles at the discrete eigenvalues of $h$. However, the presence of the factor $(1-P_{123})$ in the formulas for $A_0(z)$ and $A_{ij}(z)$ eliminates the possibility of such poles occurring.

LEMMA 5.2. *Assume the hypotheses of Theorem* 1.1 *and let* $N=4$. *Assume (for convenience) that the potentials* $V_{ij}$ *are bounded. Then the following operators are norm well behaved on* $\mathbb{C}\backslash\sigma_{\text{ess}}(H_{ijk})$:

$$\rho_{ijk}(z-H_{ijk})^{-1}(1-P_{ijk})\,\rho_{ijk}, \tag{5.5}$$

$$\rho_{ij}(1-P_{ij})(z-H_{ijk})^{-1}(1-P_{ijk})\,\rho_{ijk}, \tag{5.6}$$

$$\rho_{ij}(1-P_{ij})(1-P_{ij}-P_{ik}-P_{jk})(z-H_{ijk})^{-1}(1-P_{ijk})\,\rho_{ijk}. \tag{5.7}$$

*Proof.* As in the proof of Lemma 5.1, the operators under consideration are decomposable under a direct integral decomposition, and one may work with the individual fibers. The fibers of (5.5) are of the form $A_0(z-p^2)$, with $A_0(z)$ as in the above discussion. The fact that (5.5) is norm well behaved follows immediately from the fact that $A_0(z)$ is norm well behaved. Similarly, (5.7) is norm well behaved because $A_{ij}(z)$ is norm well behaved.

To study (5.6), we write it as

$$\rho_{ij}(1-P_{ij})(1-P_{ij}-P_{ik}-P_{jk})(z-H_{ijk})^{-1}(1-P_{ijk})\,\rho_{ijk}$$
$$+(\rho_{ij}(1-P_{ij})(P_{ik}+P_{jk})\,\rho_{ijk}^{-1})(\rho_{ijk}(z-H_{ijk})^{-1}(1-P_{ijk})\,\rho_{ijk}).$$

In this expression, the first term is just (5.7). The first factor of the second term is bounded by Lemma 2.3. The second factor is just (5.5). Consequently, the result for (5.6) follows from the result for (5.5) and (5.7). ∎

This concludes our discussion of threshold behavior except for one more or less trivial result concerning the Hamiltonian $H_{ij,kl}$. If one assumes that both $h_{ij}$ and $h_{kl}$ have no threshold resonances and no threshold eigenvalues, then $H_{ij,kl}$ has generic threshold behavior. This is summarized in the following lemma.

LEMMA 5.3. *Assume the hypotheses of Lemma* 5.2. *Then the following operators are norm well behaved on* $\mathbb{C}\backslash\sigma(H_{ij,kl})$:

$$\rho_{ij}(1-P_{ij})(z-H_{ij,kl})^{-1}(1-P_{ij,kl})\,\rho_{ij,kl} \tag{5.8}$$

$$\rho_{kl}(1-P_{kl})(z-H_{ij,kl})^{-1}(1-P_{ij,kl})\,\rho_{ij,kl} \tag{5.9}$$

$$\rho_{ij,kl}(z-H_{ij,kl})^{-1}(1-P_{ij,kl})\,\rho_{ij,kl} \tag{5.10}$$

*Proof.* Consider only (5.8) and (5.10) since (5.9) is (5.8) with permuted indices.

Since $\rho_{ij}^{-1}\rho_{ij,kl}$ is bounded, the result for (5.8) follows from that for $\rho_{ij}(1-P_{ij})(z-H_{ij,kl})^{-1}\rho_{ij}$. This operator commutes with $(H_{ij,kl}-h_{ij})=\Lambda$. Since $\Lambda$ is self-adjoint, the spectral theorem shows that it may be represented as multiplication by $\lambda$ in some representation. In this representation our operator is fibered with fibers $\rho_{ij}(1-P_{ij})(z-\lambda-h_{ij})^{-1}\rho_{ij}$, where $\lambda \in [\inf \sigma(h_{k,l}), \infty)$.

Using this and the argument used to prove (5.1) is norm well behaved, we see that (5.8) is norm well behaved.

The operator (5.10) can be written as

$$\rho_{ij,kl}(z-H_{ij,kl})^{-1}(1-P_{ij})(1-P_{ij,kl})\,\rho_{ij,kl} \\ + \rho_{ij,kl}(z-H_{ij,kl})^{-1}P_{ij}(1-P_{ij,kl})\,\rho_{ij,kl}.$$

The techniques used to study (5.8) control the first term. In the second term we note that $P_{ij}(1-P_{ij,kl})=P_{ij}(1-P_{kl})$. After replacing $P_{ij}(1-P_{ij,kl})$ with $P_{ij}(1-P_{kl})$, we may use the techniques which control (5.9) to control the second term. ∎

## 6. Verification of Subsystem Resolvent Estimates

In this section we show that hypotheses (a), (b), (c), (d), and (e) of Proposition 4.1 follow from the hypotheses of Theorem 1.1. By doing so we complete the proof of Theorem 1.1.

Hypothesis (a) of Proposition 4.1 follows from the hypotheses of Theorem 1.1 by Lemmas 5.1, 5.2, and 5.3. Hypothesis (e) of Proposition 4.1 follows from the hypotheses of Theorem 1.1 by Theorem 8.1 of [18]. Hypotheses (c) and (d) of Proposition 4.1 follow from the hypotheses of Theorem 1.1 by Lemmas V.8 and V.9 of [10]. (One does not need all the machinery of [10] to prove these two lemmas, but we have nothing new to add to the proofs of these lemmas which appear in [10].) Consequently, we need only verify hypothesis (b) of Proposition 4.1.

Up to permutations of the particles, hypothesis (b) of Proposition 4.1 makes statements about only two operator valued functions. These are

$$\rho_{123}(1-P_{123})(z-H_{123})^{-1}\rho_{14} \tag{6.1}$$

$$\rho_{12,34}(1-P_{12,34})(z-H_{12,34})^{-1}\rho_{13}. \tag{6.2}$$

We need only show these two are norm well behaved.

LEMMA 6.1. *Assume the hypotheses of Theorem* 1.1 *and* (*for convenience*) *assume the potentials* $V_{ij}$ *are bounded. Then the following are norm well behaved on* $\mathbb{C}\backslash\sigma_{\text{ess}}(H)$:

$$\rho_{124}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123} \tag{6.3}$$

$$\rho_{12,34}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123}. \tag{6.4}$$

*Proof.* Consider only (6.3). The proof for (6.4) is similar. The operator (6.3) may be written as

$$\begin{aligned}&\rho_{124}(1-P_{12})(z-H_{123})^{-1}(1-P_{123})\,\rho_{123}\\&\quad+\rho_{124}P_{12}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123}.\end{aligned}$$

The first term in this expression is a bounded operator, $\rho_{124}\rho_{12}^{-1}$, times (5.6). It is norm well behaved by Lemma 5.2. The second term may be rewriten as

$$\begin{aligned}&\rho_{124}P_{12}(z-H_{12})^{-1}(1-P_{123})\,\rho_{123}\\&\quad+\rho_{124}P_{12}(z-H_{12})^{-1}(V_{13}+V_{23})(z-H_{123})^{-1}(1-P_{123})\,\rho_{123}\\&=\rho_{124}P_{12}(z-H_{12})^{-1}\rho_{123}[\rho_{123}^{-1}(1-P_{123})\,\rho_{123}]\\&\quad+\rho_{124}P_{12}(z-H_{12})^{-1}\rho_{123}[\rho_{123}^{-1}P_{12}(V_{13}+V_{23})\,\rho_{123}^{-1}]\\&\quad\times(\rho_{123}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123})\end{aligned}$$

On the right-hand side, the factors in the square brackets are bounded by Lemma 2.3. The last factor in the second term is norm well behaved since hypothesis (a) is a consequence of the hypotheses of Theorem 1.1. So, we need only prove $\rho_{124}P_{12}(z-H_{12})^{-1}\rho_{123}$ is norm well behaved. This follows from a trivial modification of Lemma II.3 of [10]. ∎

LEMMA 6.2. *Assume the hypotheses of Theorem* 1.1 *and* (*for convenience*) *assume the* $V_{ij}$ *are bounded. Then the operator valued function* (6.1) *is norm well behaved on* $\mathbb{C}\backslash\sigma_{\text{ess}}(H)$.

*Proof.* It is sufficient to show $\rho_{14}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123}$ is norm well behaved. However, this operator may be written as

$$\begin{aligned}&\rho_{14}[(P_{12}+P_{13}+P_{23})+(1-P_{12}-P_{13}-P_{23})](z-H_{123})^{-1}(1-P_{123})\,\rho_{123}\\&\quad=(\rho_{14}P_{12}\rho_{124}^{-1})(\rho_{124}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123})\\&\qquad+(\rho_{14}P_{13}\rho_{134}^{-1})(\rho_{134}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123})\\&\qquad+(\rho_{14}P_{23}\rho_{14,23}^{-1})(\rho_{14,23}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123})\\&\qquad+\rho_{14}(1-P_{12}-P_{13}-P_{23})(z-H_{123})^{-1}(1-P_{123})\,\rho_{123}.\end{aligned}$$

The first two terms on the right-hand side are products of factors which are bounded by Lemma 2.3 and factors of type (6.3). The third term is the product of a factor which is bounded by Lemma 2.3 and a factor of type (6.4). Thus, the first three terms are norm well behaved.

Letting $Q=(1-P_{13}-P_{23}-P_{12})$ we may rewrite the fourth term as

$$
\begin{aligned}
&(\rho_{14}(z-H_{23})^{-1}\rho_{123})(\rho_{123}^{-1}Q\rho_{123})\\
&\quad+(\rho_{14}(z-H_{23})^{-1}\rho_{123})(\rho_{123}^{-1}[H,Q]\,\rho_{123}^{-1})(\rho_{123}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123})\\
&\quad+(\rho_{14}(z-H_{23})^{-1}\rho_{13})(\rho_{13}^{-1}\mathrm{V}_{13}\rho_{13}^{-1})\\
&\quad\times(\rho_{13}(1-P_{13})\,Q(z-H_{123})^{-1}(1-P_{123})\,\rho_{123})\\
&\quad+(\rho_{14}(z-H_{23})^{-1}\rho_{13})(\rho_{13}^{-1}\mathrm{V}_{13})\\
&\quad\times(P_{13}(P_{12}+P_{23})\,\rho_{123}^{-1})(\rho_{123}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123})\\
&\quad+(\rho_{14}(z-H_{23})^{-1}\rho_{12})(\rho_{12}^{-1}\mathrm{V}_{12}\rho_{12}^{-1})\\
&\quad\times(\rho_{12}(1-P_{12})\,Q(z-H_{123})^{-1}(1-P_{123})\,\rho_{123})\\
&\quad+(\rho_{14}(z-H_{23})^{-1}\rho_{12})(\rho_{12}^{-1}\mathrm{V}_{12})(P_{12}(P_{13}+P_{23})\,\rho_{123}^{-1})\\
&\quad\times(\rho_{123}(z-H_{123})^{-1}(1-P_{123})\,\rho_{123}).
\end{aligned}
$$

All of the $z$-independent factors which occur here are bounded. The first factor in every term is norm well behaved by trivial modifications of Lemma II.3 of [10]. The remaining factors are of the forms (5.5) or (5.7), and are norm well behaved by Lemma 5.2. ∎

LEMMA 6.3. *Assume the hypotheses of Theorem* 1.1 *and assume* (*for convenience*) *that the potentials* $\mathrm{V}_{ij}$ *are bounded. Then the operator* (6.2) *is norm well behaved on* $\mathbb{C}\backslash\sigma_{\mathrm{ess}}(H)$.

*Proof.* It is sufficient to show $\rho_{13}(z-H_{12,34})^{-1}(1-P_{12,34})\,\rho_{12,34}$ is norm well behaved. We can rewrite this operator as

$$
\begin{aligned}
&\rho_{13}(P_{34}+(1-P_{34}))(z-H_{12,34})^{-1}(1-P_{12,34})\,\rho_{12,34}\\
&\quad=\rho_{13}P_{34}(z-H_{34})^{-1}(1-P_{12,34})\,\rho_{12,34}\\
&\qquad+(\rho_{13}P_{34}(z-H_{34})^{-1}\rho_{12})(\rho_{12}^{-1}P_{34}\mathrm{V}_{12}\rho_{12,34}^{-1})\\
&\qquad\times(\rho_{12,34}(z-H_{12,34})^{-1}(1-P_{12,34})\,\rho_{12,34})\\
&\qquad+\rho_{13}(z-H_{12})^{-1}(1-P_{34})(1-P_{12,34})\,\rho_{12,34}\\
&\qquad+(\rho_{13}(z-H_{12})^{-1}\rho_{34})(\rho_{34}^{-1}\mathrm{V}_{34}\rho_{34}^{-1})\\
&\qquad\times(\rho_{34}(1-P_{34})(z-H_{12,34})(1-P_{12,34})\,\rho_{12,34}).
\end{aligned}
$$

The $z$-independent factors on the right-hand side are bounded. The first and third terms and the initial factors of the second and fourth terms are now well behaved by trivial modifications of the proof of Lemma II.3 of [10] and Lemma 2.3. The final factors in the second and fourth terms are norm well behaved by Lemma 5.3. ▮

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